Atomic Entanglement vs Photonic Visibility for Quantum Criticality of Hybrid System

M.X. Huo[1], Ying Li[1], Z. Song[1],* and C.P. Sun[2†]
[1]*Department of Physics, Nankai University, Tianjin 300071, China and*
[2]*Institute of Theoretical Physics, Chinese Academy of Sciences, Beijing, 100080, China*

To characterize the novel quantum phase transition for a hybrid system consisting of an array of coupled cavities and two-level atoms doped in each cavity, we study the atomic entanglement and photonic visibility in comparison with the quantum fluctuation of total excitations. Analytical and numerical simulation results show the happen of quantum critical phenomenon similar to the Mott insulator to superfluid transition. Here, the contour lines respectively representing the atomic entanglement, photonic visibility and excitation variance in the phase diagram are consistent in the vicinity of the non-analytic locus of atomic concurrences.

PACS numbers: 03.65.Ud, 42.50.Dv, 73.43.Nq, 03.67.-a

Introduction: It is crucial in the modern theory of second order phase transitions to introduce 'order parameter', whose non-vanishing average value characterizes one or more phases and usually breaks a symmetry of the Hamiltonian. But for 'quantum' phase transitions on the behavior of matter near zero temperature [1], it is more subtle to use the traditional Ginzburg-Landau-Wilson paradigm, since in some cases the natural description of the quantum criticality is not based on the order parameter [2].

Actually, for some systems with complicated structures it might be difficult to chose an appropriate order parameter to correctly characterize the emergent phenomena. The purpose of this paper with a specific example is to demonstrate that, though we can not make sure what is the appropriate order parameter for a photon-atom hybrid system, some physical observable quantities can be used to witness its quantum critical phenomenon.

The hybrid system we consider is a coupled waveguide resonator array (CWRA) where each cavity is doped with a two-level atom (see Fig. 1). This hybrid architecture was suggested as a quantum coherent device to transfer and store quantum information as well as to create the laser-like output [3, 4, 5]. As for the quantum phase transition, it is observed that such a doped CWRA can simulate the Mott like transition of light from "the Mott insulator (MI) to superfluid (SF)" [6] since a doped atom can induce the effective photon-photon interaction in each cavity. Together with the inter-cavity hopping of localized phonons, this nonlinear photon-photon coupling can result in the Bose-Hubbard model for Mott phase transition [7]. Recent experiments [8] using cold atoms in an optical lattice have clearly demonstrated the quantum phase transition predicted by the Bose-Hubbard model. Actually the Bose-Hubbard theory of Mott phase transition for cold atoms [9] is also based on the assumption of the order parameter, the average of the annihilation operator of boson in each site. In mean field approach, the average of the annihilation operator of boson is usually employed, while "number variance" is used in many other methods to discriminate Mott insulating and superfluid phases [10, 11].

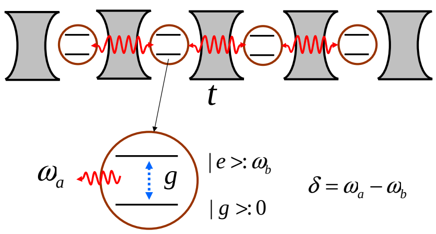

FIG. 1: *Schematic setup of a cavity array with each one containing a two-level atom. Photons of mode ω_a can tunnel between adjacent cavities with hopping integral t and couple to the atoms with strength g. This atom-photon lattice is expected to simulate Mott insulator and superfluid transition.*

For the hybrid system, the study of quantum phase transition of light in Ref. [6] still assumes the same order parameter in terms of photons, while more strict assumption of order parameter [10, 11] was implicitly made in terms of the number of the polariton, which is a mixture of photon and atom. Here, we will not adopt such local order parameters of quasi-particle as direct characterizations of quantum phase transition, but pay attention to some observable quantities to characterize the critical phenomenon of a hybrid system. To this end we make use of atomic entanglement from the point view of quantum information, as well as photon visibility in terms of quantum optical theory. We remark that, as a quantum nonlocal property, the quantum entanglement plays an important role in the study of quantum phase transitions [12, 13, 14, 15, 16]. We examine the signature of the Mott insulator to superfluid transition, the excitation number variance, and other two observable quantities, concurrence between two atoms and the visibility of photons, in lattice atom-photon hybrid systems of small size by analytical and numerical methods respectively. Our results reveal nontrivial connections among the three quantities in such an intriguing way: contour lines of three quantities in the phase diagram are approximately consistent with each other when the non-analyticity of concurrences occurs. It firmly shows that such three quantities are signatures for the MI to SF transition in such a atom-photon hybrid system.

Model setup and the quasi-excitation fluctuation. We consider an array of N coupled cavities with each one containing a single two-level atom [3, 4, 5, 11]. The photon transmission of cavity-to-cavity occurs as a hopping mechanism if there were the interaction between atom and cavity mode. Such hybrid system can be implemented with the defect array in photonic crystal [17] or Josephson junction array in cavity [3]. The Hamiltonian of a hybrid system, or a lattice atom-photon system, $H = H_{free} + H_{int} + H_{hop}$ is decomposed as three parts, free Hamiltonians of light and atom,

$$H_{free} = \omega_a \sum_{i=1}^{N} a_i^\dagger a_i + \omega_b \sum_{i=1}^{N} |e\rangle_i \langle e|, \tag{1}$$

the cavity-mode-atom interaction in the ith defect

$$H_{int} = g \sum_{i=1}^{N} \left(a_i^\dagger |g\rangle_i \langle e| + \text{H.c.} \right), \tag{2}$$

with strength g and the photon hopping between NN defects

$$H_{hop} = -t \sum_{i=1}^{N} \left(a_i^\dagger a_{i+1} + \text{H.c.} \right), \tag{3}$$

with hopping integral constant t for the tunneling between adjacent cavities. Here, $|g\rangle_i$ ($|e\rangle_i$) denotes the ground (excited) state of the atom placed at ith cavity; $a_i^\dagger$ and a_i are the creation and annihilation operators of a photon at defect i. Obviously the total excitation number

$$\hat{\mathcal{N}} = \sum_{i=1} \hat{\mathcal{N}}_i = \sum_{i=1} \left(a_i^\dagger a_i + S_i^z + \frac{1}{2} \right) \tag{4}$$

is conserved quantity for the Hamiltonian H, i.e., $[H, \hat{\mathcal{N}}] = 0$, where $2S_i^z |e\rangle_i = |e\rangle_i$ and $2S_i^z |g\rangle_i = -|g\rangle_i$.

It can be seen that $\hat{\mathcal{N}}$ is just the single excitation number of the polaritons. It is well known that the conventional Mott insulator to superfluid phase transition occurs in a Bose-Hubbard model. Here, when the repulsive interaction between bosons is large enough in the Mott phase, the number fluctuation would become energetically unfavorable, forcing the system into a number state and exhibiting vanishing particle number fluctuation. In the superfluid regime, atoms are delocalized with non-vanishing particle number fluctuation. As for the present hybrid system, the fundamental excitations are polaritons [11] and the mechanism of the Mott transition is due to the effect of photon blockade. Since the photon number is not conserved in such system, the photon number fluctuation $\Delta n_i = \Delta(a_i^\dagger a_i)$ is not appropriate to characterize the superfluid phase as that for a pure Bose-Hubbard model. This is because Δn_i does not vanish even in the Mott insulator regime due to the couplings between photons and atoms. Hereafter, we define the variance ΔA by $(\Delta A)^2 = \left\langle (A)^2 \right\rangle - \langle A \rangle^2$. Therefore, one can take the excitation number fluctuation per site $\Delta\mathcal{N}_i$ as an order parameter to characterize the Mott transition. In the large detuning limit $\delta = \omega_a - \omega_b \gg 0$, all atoms are in excited states, which is perfectly number squeezed states, i.e., $\Delta\mathcal{N}_i = 0$ for all sites. In the other limit $\delta \ll 0$, all atoms are in ground states. Obviously two-atom concurrence vanishes and the density fluctuation becomes $\Delta\mathcal{N}_i = \sqrt{\langle a_i^\dagger a_i^\dagger a_i a_i \rangle} = \sqrt{(N-1)/N} \simeq 1$ since $\mathcal{N} = N$ in this case.

Atomic entanglement characterized by concurrence. Intuitively, two atoms in two adjacent cavities should entangle with each other due to the hopping of phonon from one cavity to another. Now we try to describe this kind of atomic entanglement induced by coupled photons. Obviously, if the photon is in quantum phase transition, the critical photon induced atomic entanglers can characterize this critical behavior.

We express the concurrence characterizing quantum entanglement in terms of observable quantities such as correlation functions. The complete basis vectors of the total system are denoted by

$$|\{n_j, s_j\}\rangle = |n_1, .., n_N; s_1, .., s_N\rangle = \prod_{j=1}^{N} |n_j\rangle \otimes |s_j\rangle \tag{5}$$

where $|n_j\rangle$ is the Fock state of photon and $|s_j\rangle = |g\rangle_i, |e\rangle_i$ for $s_j = 0, 1$ respectively. The fact that $\hat{\mathcal{N}}$ is conserved can be reflected by the matrix element vanishing of the density operator $\rho = \rho(H)$ on the above basis for any state of the hybrid system, that is,

$$\rho_{\{n_j, s_j\}}^{\{n'_j, s'_j\}} = \rho_{\{n_j, s_j\}}^{\{n_j, s_j\}} \delta[\sum (n_j + s_j - n'_j - s'_j)] \tag{6}$$

The functional $\rho(H)$ of the Hamiltonian may be a ground state or thermal equilibrium states. The reduced density matrix $\rho^{(12)} = Tr_p Tr_{3..N}^s[\rho(H)]$ for two atomic quasi-spins, e.g., s_1 and s_2 are obtained as

$$\begin{aligned} [\rho^{(12)}]_{s'_1 s'_2, s_1 s_2} &= \sum_{[n_j; s_3 \ldots s_N]} \rho_{n_j, s_1 s_2 s_3 \ldots s_N}^{n_j, s'_1 s'_2 s_3 .. s_N} \delta[\sum (s_j - s'_j)] \\ &= \delta(s_1 + s_2 - s'_1 - s'_2) \sum_{[n_j; s_3 \ldots s_N]} \rho_{n_j, s_1 s_2 s_3 .. s_N}^{n_j, s'_1 s'_2 s_3 .. s_N} \end{aligned} \tag{7}$$

by tracing over all photon variables (with Tr_p) and atomic variables except for s_1 and s_2. The corresponding reduced density matrix for two atoms i and j is of the form

$$\rho^{(ij)} = \begin{pmatrix} u_{ij}^{+} & 0 & 0 & 0 \\ 0 & w_{ij}^{1} & z_{ij}^{*} & 0 \\ 0 & z_{ij} & w_{ij}^{2} & 0 \\ 0 & 0 & 0 & u_{ij}^{-} \end{pmatrix}. \tag{8}$$

According to Refs. [18, 19], the concurrence $C_{ij} = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}$ shared between two atoms i and j is obtained in terms of the the square roots $\{\lambda_i\}$ ($\lambda_1 = \max\{\lambda_i\}$) of eigenvalues of the non-Hermitian matrix $\rho\widetilde{\rho}$. Here $\widetilde{\rho} = (\sigma_y \otimes \sigma_y)\,\rho^*\,(\sigma_y \otimes \sigma_y)$. Using the observable quantities, the quantum correlation $z_{ij} = \langle\psi|\, S_i^+ S_j^- \,|\psi\rangle$, $u_{ij}^{\pm} = \langle\psi|\,(1/2 \pm S_i^z)\,(1/2 \pm S_j^z)\,|\psi\rangle$, the concurrence is rewritten as a computable form

$$C_{ij} = 2\max(0, |z_{ij}| - \sqrt{u_{ij}^{+} u_{ij}^{-}}). \tag{9}$$

We note that this formula for the concurrence of two quasi-spin in a hybrid system is the same as that for pure spin-1/2 system [18, 19]. The non-analyticity of concurrence arises from the abrupt switch of the sign of quantity $|z_{ij}| - \sqrt{u_{ij}^{+} u_{ij}^{-}}$ and can be used to determine quantum phase transitions.

Photon visibility in hybrid system. Similar to the transition of superfluid to Mott insulator in Bose-Hubbard model [20], two phases of the atom-photon hybrid system can also be delimited through the quantum coherence of the ground state. In Mott insulating phase, the quantum coherence of photons is completely destroyed due to the photon blockade. In superfluid phase limit, the quantum coherence of photons gets its maximum. Therefore the quantum coherence of photons can be employed to indicate phases, which is characterized by a observable quantity, the visibility of 'interference fringes'

$$\mathcal{V} = \frac{V_{\max} - V_{\min}}{V_{\max} + V_{\min}}. \tag{10}$$

Here, $V_{\max}$ and $V_{\min}$ are the maximum and minimum of the photon number distribution of ground state in the k space

$$V(k) = \frac{1}{N} \sum_{j,l} e^{ik(j-l)} \left\langle a_j^{\dagger} a_l \right\rangle. \tag{11}$$

In the strong photon blockade limit, $\mathcal{V} = 0$ while in the superfluid limit, $\mathcal{V} = 1$. Comparing with the local quantity, the photon number fluctuation, the visibility is more appropriate to discriminate two phases.

Characterizing quantum criticality. For the lattice atom-photon system, we now consider the connections among three quantities $\Delta\mathcal{N}_i$, C_{ij}, and $\mathcal{V}$ around the critical point.

We start with an extreme case $g = 0$. At zero temperature, the physics of the lattice atom-photon model can be described in two regimes separated by the boundary $\delta = 2t$. In the region $\delta > 2t$ ($\delta < 2t$), the ground state is in typical Mott insulating (superfluid) phase with $\Delta\mathcal{N}_i = 0$ ($\sqrt{(N-1)/N}$), $\mathcal{V} = 0$ (1), and $C_{ij} = 0$ (0), respectively. At line $\delta = 2t$, the model admits multifold degenerate ground states with energy $\varepsilon^{(0)} = N\omega_b$, and the excitation number fluctuation and visibility experience a big jump, while the concurrence between two atoms is 'uncertainty' due to the energy-level crossing.

When the atom-photon interaction g is switched on, it becomes avoided level crossing. This fact will result in the quantum fluctuation driving the transition from Mott insulator to superfluid phase, which corresponds to the non-vanishing concurrence between atoms. To illustrate this mechanism quantitatively, we just switch on atom-photon couplings in cavities i and j and leave all other coupling to be zero. For very small g, the un-perturbable ground states

$$\begin{aligned} |\phi_1\rangle &= |N\rangle_{k=0} \prod_l |g\rangle_l \,, |\phi_2\rangle = |N-1\rangle_{k=0} |e\rangle_i |G_i\rangle, \\ |\phi_3\rangle &= |N-1\rangle_{k=0} |e\rangle_j |G_j\rangle, \\ |\phi_4\rangle &= |N-2\rangle_{k=0} |e\rangle_i |e\rangle_j |G_{i,j}\rangle, \end{aligned} \tag{12}$$

are degenerate, where $|n\rangle_k$ denotes the photon Fock state in k space and $|G_{i,j,..}\rangle = \prod_{l \neq i,j,..} |g\rangle_l$ denotes the atomic state of all atoms except $l = i, j, ...$ Up to the first order perturbation with energy correction $\varepsilon^{(1)} = -g\sqrt{2N(1+\beta^2)} = -g\sqrt{N}\eta^{-1}$ where $\beta = \sqrt{(N-1)/N}$, the perturbed ground state is

$$|\psi_g\rangle = \eta(|\phi_1\rangle + \beta\,|\phi_4\rangle) - \frac{1}{2}(|\phi_2\rangle + |\phi_3\rangle). \tag{13}$$

The corresponding concurrence can be calculated as

$$C_{ij} = \frac{(1-\beta)^2}{2(1+\beta^2)}. \tag{14}$$

As δ being apart from the degenerate point, the concurrence C_{ij} decreases due to the energy competition of two phases. Therefore, this heuristic analysis has shown the simple relation among concurrence, visibility, and excitation number fluctuation around quantum phase transition critical point: the excitation number fluctuation and visibility both exhibit an abrupt drop while the concurrence has a sharp maximum. It can be predicted that as g increases, changes of the three quantities will be slow due to the strongly coupling between atoms and photons. In the following, it will be investigated for small system in wide range of parameters by numerical simulations.

We investigate three quantities in a small size system by exact diagonalization method. For open chain cavity array system, the visibility $\mathcal{V}$ can be calculated by $S(k) = 2/(N+1)\sum_{i,j} \sin(ki)\sin(kj) \left\langle a_i^{\dagger} a_j \right\rangle$, where $k = n\pi/(N+1)$, $n \in [1, N]$, while the concurrence and excitation number fluctuation can be characterized as average concurrence $\overline{C} = (1/N)\sum_{i<j} C_{ij}$ and average excitation number fluctuation $\overline{\Delta\mathcal{N}} = 1/N\sum_i \Delta\mathcal{N}_i$. In Fig.

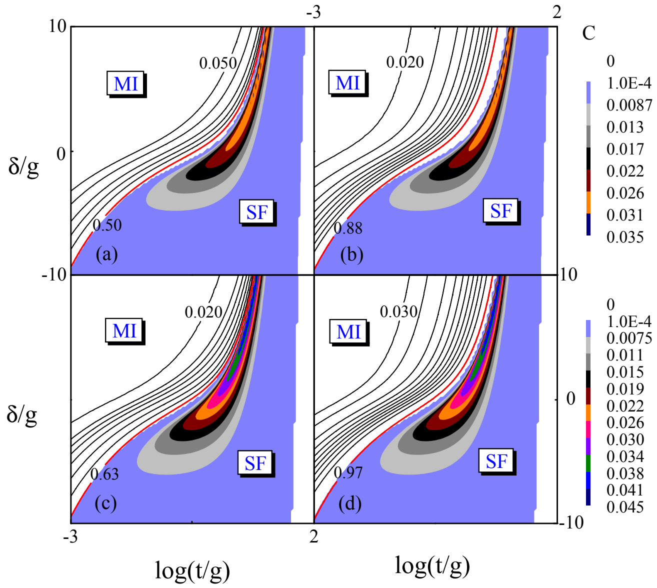

FIG. 2: *(Color online) Contours of three quantities, $\overline{\Delta\mathcal{N}}$ (dark lines in (a,c)), $\mathcal{V}$ (dark lines in (b,d)), and $\overline{C}$ (color maps in (a-d) obtained by exact diagonalization for 2- (a,b), 4- (c,d) cavity systems. Red lines in (a-d) denote closer contour lines of $\mathcal{V}$ and $\overline{\Delta\mathcal{N}}$ to the non-analytical curve of $\overline{C}$. It shows that contour lines of three quantities are consistent in the vicinity of the non-analytic locus, at which the non-analyticity of concurrence occurs.*

2, contours of three quantities obtained by exact diagonalization are plotted in the δ/g-t/g plane for 2- (Fig. 2(a, b)), 4- (Fig. 2(c, d)) cavity systems. Contours of excitation number fluctuation $\overline{\Delta\mathcal{N}}$ (dark lines in Fig. 2(a, c)) and visibility of photons $\mathcal{V}$ (dark lines in Fig. 2(b, d)) are compared with the concurrence $\overline{C}$ (color maps in Fig. 2(a-d)) as functions of the scaled detuning δ/g and photon hopping integral t/g. We see that contour lines of three quantities are consistent in the vicinity of the locus at which the non-analyticity of concurrence occurs. The non-analytic locus in δ/g-t/g plane is defined by the equation $|z_{ij}| - \sqrt{u_{ij}^{+}u_{ij}^{-}} = 0$. Red lines in Fig. 2(a-d) denote closer contour lines of $\mathcal{V}$ and $\overline{\Delta\mathcal{N}}$ to the non-analytical curve of $\overline{C}$. It also shows that the visibility and excitation number fluctuation start to drop at the non-analytic locus of concurrence. There is a slight difference between profiles of 2 and 4-cavity systems. The red contour line of $\mathcal{V}$ in 4-cavity system is closer to the non-analytic locus of $\overline{C}$ than that in 2-cavity system. It indicates that contour lines of three quantities will cover at the vicinity the non-analytic locus of $\overline{C}$ in thermodynamics limit.

Summary: In summary, we have investigated the Mott insular to superfluid transition in a hybrid system consisting of an array of coupled cavities doped with two level atoms. We investigate two non-local observable quantities, the concurrence between atoms and the visibility of photons, in comparison with the local order parameter, excitation number fluctuation, for the Mott insulator to superfluid transition. It can be observed form analytical and numerical simulation results that the atomic entanglement and photonic visibility in the phase diagram indeed can reflect the quantum critical phenomenon signatured by the total excitation variance. In principle, such non-local observable quantities of the hybrid system can be used to detect the critical point in experiment.

This work is supported by the NSFC with grant Nos. 90203018, 10474104 and 60433050, and NFRPC with Nos. 2006CB921206 and 2005CB724508.

* Electronic address: songtc@nankai.edu.cn
† Electronic address: suncp@itp.ac.cn; URL: `http://www.itp.ac.cn/~suncp`

[1] S. Sachdev, Quantum Phase Transitions (Cambridge University Press, Cambridge, England, 1999).
[2] T. Senthil, A. Vishwanath, L. Balents, S. Sachdev, M. P. A. Fisher Science **303**, 1490 (2004).
[3] L. Zhou, Y.B. Gao, Z. Song, and C.P. Sun, cond-mat/0608577, submitted to Phys. Rev. Lett..
[4] F.M. Hu, L. Zhou, T. Shi, C.P. Sun, quant-ph/0610250(2006) submitted to Phys. Rev. E.
[5] L. Zhou, J. Lu, C.P. Sun, quant-ph/0611159, submitted to Phys. Rev. A.
[6] A.D. Greentree, C. Tahan, J.H. Cole, and L.C.L. Hollenberg, Nature Phys. **2**, 856 (2006).
[7] M.P.A. Fisher, P.B. Weichman, G. Grinstein, and D.S. Fisher, Phys. Rev. B **40**, 546 (1989).
[8] M. Greiner, O. Mandel, T. Esslinger, T.W. Hänsch, and I. Bloch, Nature (London) **415**, 39 (2002).
[9] D. Jaksch, C. Bruder, J.I. Cirac, C.W. Gardiner, and P. Zoller, Phys. Rev. Lett. **81**, 3108 (1998).
[10] M.J. Hartmann, F.G.S.L. Brandao and M.B. Plenio, Nature Phys. **2**, 849 (2006).
[11] D.G. Angelakis, M.F. Santos, and S. Bose, quant-ph/0606159.
[12] A. Osterloh, L. Amico, G. Falci, and R. Fazio, Nature **416**, 608 (2002).
[13] T.J. Osborne and M. A. Nielsen, Phys. Rev. A **66**, 032110 (2002).
[14] S.J. Gu, H.Q. Lin, and Y.Q. Li, Phys. Rev. A **68**, 042330 (2003).
[15] Y. Chen, Z.D. Wang and F.C. Zhang, Phys. Rev. B **73**, 224414 (2006).
[16] X.F. Qian, Ying Li, Yong Li, Z. Song, and C.P. Sun, Phys. Rev. A **72**, 062329 (2005).
[17] M.F. Yanik, S. Fan, Phys. Rev. Lett. **92**, 083901 (2004); Q. Xu, S. Sandhu, M.L. Povinelli, J. Shakya, S. Fan, and M. Lipson, Phys. Rev. Lett. **96**, 123901 (2006).
[18] W.K. Wootters, Phys. Rev. Lett. **80**, 2245 (1998).
[19] X. Wang and P. Zanardi, Phys. Lett. A **301**, 1 (2002); X. Wang, Phys. Rev. A **66**, 034302 (2002).
[20] F. Gerbier, A. Widera, S. Fölling, O. Mandel, T Gericke, and I. Bloch, Phys. Rev. Lett. **95**, 050404 (2005); Phys. Rev. A **72**, 053606 (2005).